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On curve and surface stretching in isotropic turbulent flow

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Cocke (1969) showed that, on average, infinitesimal material lines and surfaces are stretched in incompressible isotropic turbulence. We have extended those results to obtain upper and lower bounds for the stretching of such infinitesimal elements in terms of the eigenvalues of the Green deformation tensor. These bounds are in turn used to find bounds for the stretching of finite material lines and surfaces.

1. Introduction

Cocke (1969) proved that in any incompressible flow that is statistically isotropic, the expected value of the change of the logarithm of line element lengths and surface element areas is greater than zero. Orszag (1970) noted that Cocke's result, combined with the generalized arithmetic/geometric-mean inequality, implies that the expected value of the change in the mean-square length of line elements and the area of surface elements is also greater than zero, and provided a simple proof of that result. It should be noted that these results are purely kinematical, the dynamics of fluid motion does not enter. These works, and their historical background, are described by Monin & Yaglom (1975, p. 578). This work complements the work of Cocke (1969) and Orszag (1970). We shall show, first, that their results can be improved to obtain 'tight' upper and lower bounds for the ensemble-average stretching of material line and surface elements. Then we extend the results to arbitrary curves and surfaces, and obtain upper bounds for moments of the relative dispersion of two fluid particles in terms of their initial separation.

Throughout this work we shall use $\boldsymbol{x}(\boldsymbol{a},t)$ as the Lagrangian representation of the flow, i.e. the trajectory followed by the particle that is at position $\boldsymbol{a}$ at initial time t_0, and assume that it is smooth enough in a space–time region and is an invertible mapping for fixed t so that our manipulations are legitimate. Finally for an arbitrary vector $\boldsymbol{v}$, with elements $v_i (i=1,2,3)$, v or $|\boldsymbol{v}|$ is its magnitude, $\boldsymbol{v}_{,j} = \partial \boldsymbol{v}/\partial a_j$ or $v_{i,j} = \partial v_i/\partial a_j$ is its gradient.

2. Line stretching

The motion of an infinitesimal material line element $\boldsymbol{l}$ is governed by

$$\frac{\mathrm{d}\boldsymbol{l}}{\mathrm{d}t} = (\boldsymbol{l}\cdot\nabla)\boldsymbol{u} \quad \text{or} \quad \frac{\mathrm{d}l_i}{\mathrm{d}t} = \frac{\partial u_i}{\partial x_j} l_j \tag{1}$$

(following the notation of Monin & Yaglom 1975), with the initial condition $\boldsymbol{l}(t_0) = \boldsymbol{l}_0$. Here $\mathrm{d}/\mathrm{d}t$ is the material derivative following the motion. The solution is given directly by the flow map $\boldsymbol{x}(\boldsymbol{a},t)$ as $l_i = x_{i,j}\,l_{0j}$, where $\boldsymbol{x}(\boldsymbol{a},t_0) = \boldsymbol{a}$. The Jacobian of the flow map

$$J \equiv \det(x_{i,j}) = 1 \tag{2}$$

when the flow is incompressible, see Batchelor (1977, p. 79). The length of a line element is given by

$$l^2 = l_i\,l_i = W_{jk}\,l_{0j}\,l_{0k} \tag{3}$$

where the Green deformation tensor $W_{jk} = x_{i,j}\,x_{i,k}$ is a symmetric positive definite matrix which has real, strictly positive eigenvalues w_i and determinant

$$\det(\boldsymbol{W}) = w_1\,w_2\,w_3 = \det(x_{i,j})^2 = J^2. \tag{4}$$

Let $\boldsymbol{A} = (a_{ij})$ be the unitary (rotation) matrix corresponding to diagonalization of $\boldsymbol{W}$. Then $|\boldsymbol{A}\boldsymbol{l}_0| = l_0$ and (3) becomes

$$\begin{aligned}\frac{l^2}{l_0^2} &= w_1\frac{(\boldsymbol{A}\boldsymbol{l}_0)_1^2}{|\boldsymbol{A}\boldsymbol{l}_0|^2}+w_2\frac{(\boldsymbol{A}\boldsymbol{l}_0)_2^2}{|\boldsymbol{A}\boldsymbol{l}_0|^2}+w_3\frac{(\boldsymbol{A}\boldsymbol{l}_0)_3^2}{|\boldsymbol{A}\boldsymbol{l}_0|^2},\\ &= w_1\sin^2\theta\cos^2\psi+w_2\sin^2\theta\sin^2\psi+w_3\cos^2\theta,\end{aligned} \tag{5}$$

where θ and ψ are spherical polar angles of the vector $\boldsymbol{l}_0$ relative to the principal axes of $\boldsymbol{W}$. The polar axis of the system is taken along the eigenvector corresponding to w_3.

Equation (5) gives the square of the strain experienced by a single material line element in one realization of a flow. Following Cocke and Orszag, we shall use (5), and the assumption of statistical isotropy, to determine bounds for fluid element stretching in isotropic turbulence. To do so, we need to define the ensemble over which the average is to be taken, and to make the appropriate assumptions about the joint probability distribution of the variables involved, namely w_i, θ and ψ.

In general, the ensemble consists of a large collection of events, each of which is the deformation of a single fluid element at a single flow point. Both Cocke and Orszag consider the deformation of a *fixed* initial line element by an ensemble of isotropic flows. In that case the deformations are obviously statistically independent of the line elements upon which they act, and the probability density is uniform over the unit sphere ($0 \leqslant \theta \leqslant \pi, 0 \leqslant \psi \leqslant 2\pi$) and depends only on w_i. Ensemble averages over θ and ψ then reduce to averages over the unit sphere, which has the infinitesimal area $\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\psi$ and total area 4π. Their results remain valid for any ensemble of initial line elements, whether isotropic or not, so long as the elements are independent of the deformations, and the deformations are statistically isotropic. Conversely, the probability density is also uniform over the unit sphere when the ensemble consists of a *fixed* deformation acting on an isotropic distribution of initial line elements, and therefore the results are also valid for anisotropic flows if the ensemble of line elements is statistically isotropic and independent of the deformations. The crucial assumption is the statistical independence of the deformations and the initial line elements. When the line elements and deformations are not independent the distribution will, in general, not be uniform even if the deformations and/or the line elements are isotropic with respect to a fixed frame. It is not clear, for example, that the average stretching of vortex lines in a fluid of low viscosity can be treated under this assumption. In general *dynamically significant* initial material lines cannot be assumed *a priori* to be independent of the deformation tensor. Cocke applies an inequality to the logarithm (a concave function) of (5) and integrates over the unit sphere to arrive at his result, while Orszag simply integrates (5) over the unit sphere and applies an inequality to reach his result. We shall also use (5), and inequalities appropriate to various convex and concave functions to obtain more precise bounds for the average stretching.

A lower bound can be found by considering the concave function $\phi = x^{\frac{1}{2}}$ and, noting that the coefficients of w_i in (5) sum to one, applying to special case of Jensen's inequality (see the Appendix, equation (A 2)) to (5):

$$\frac{l}{l_0} \geqslant w_1^{\frac{1}{2}} \sin^2\theta\cos^2\psi + w_2^{\frac{1}{2}}\sin^2\theta\sin^2\psi + w_3^{\frac{1}{2}}\cos^2\theta. \tag{6}$$

It is clear from (5) directly that

$$\frac{l}{l_0} \leqslant w_1^{\frac{1}{2}}|\sin\theta\cos\psi| + w_2^{\frac{1}{2}}|\sin\theta\sin\psi| + w_3^{\frac{1}{2}}|\cos\theta|. \tag{7}$$

The averages over the unit sphere of the coefficients of $w_i^{\frac{1}{2}}$ in (6) and (7) are $\frac{1}{3}$ and $\frac{1}{2}$ respectively and give the bounds

$$\tfrac{1}{3}\langle w_1^{\frac{1}{2}} + w_2^{\frac{1}{2}} + w_3^{\frac{1}{2}}\rangle \leqslant \left\langle \frac{l}{l_0}\right\rangle \leqslant \tfrac{1}{2}\langle w_1^{\frac{1}{2}} + w_2^{\frac{1}{2}} + w_3^{\frac{1}{2}}\rangle. \tag{8}$$

We can also find bounds in terms of the sum of the eigenvalues:

$$\tfrac{1}{2}\langle (w_1+w_2+w_3)^{\frac{1}{2}}\rangle \leqslant \left\langle \frac{l}{l_0}\right\rangle \leqslant \langle (w_1+w_2+w_3)^{\frac{1}{2}}\rangle. \tag{9}$$

The upper bound follows immediately since the coefficients of w_i in (5) are bounded by one, and the lower bound is found by again using the concave function $\phi = x^{\frac{1}{2}}$ and (A 2) to give

$$\begin{aligned} \frac{l}{l_0} &= (w_1+w_2+w_3)^{\frac{1}{2}}\left(\frac{w_1\sin^2\theta\cos^2\psi + w_2\sin^2\theta\sin^2\psi + w_3\cos^2\theta}{w_1+w_2+w_3}\right)^{\frac{1}{2}} \\ &\geqslant (w_1+w_2+w_3)^{\frac{1}{2}}\frac{w_1|\sin\theta\cos\psi| + w_2|\sin\theta\sin\psi| + w_3|\cos\theta|}{w_1+w_2+w_3}. \end{aligned} \tag{10}$$

The lower bound follows because the average over the unit sphere of the coefficients of w_i in the inequality (10) is again $\frac{1}{2}$.

We then obtain, by the same methods, bounds for the moments of l/l_0 in terms of moments of either the individual w_i, or their sum. This can be accomplished simply by the above method and/or using the inequalities $\frac{1}{3}(a^r+b^r+c^r) \leqslant (a+b+c)^r \leqslant 3^r(a^r+b^r+c^r)$ for $a, b, c, r \geqslant 0$. Examples of these estimates which are sharp and symmetric are

$$\left\langle \frac{w_1^{p/2}+w_2^{p/2}+w_3^{p/2}}{3}\right\rangle \leqslant \left\langle \frac{l^p}{l_0^p}\right\rangle \leqslant \left\langle \frac{w_1+w_2+w_3}{3}\right\rangle^{p/2} \quad \text{for} \quad 0 < p \leqslant 2, \tag{11}$$

$$\left\langle \frac{l^2}{l_0^2}\right\rangle = \tfrac{1}{3}\langle w_1+w_2+w_3\rangle, \tag{12}$$

$$\left\langle \frac{w_1+w_2+w_3}{3}\right\rangle^{p/2} \leqslant \left\langle \frac{l^p}{l_0^p}\right\rangle \leqslant \left\langle \frac{w_1^{p/2}+w_2^{p/2}+w_3^{p/2}}{3}\right\rangle \quad \text{for} \quad 2 \leqslant p < \infty. \tag{13}$$

Orszag's result (12) follows immediately by averaging (5) over the unit sphere. The lower bound in (11) and the upper bound in (13) follows from (A 2) using $\phi = x^{\frac{p}{2}}$. The upper bound in (11) and the lower bound in (13) are a consequence of raising both sides of (12) to the power $\frac{1}{2}p$ and applying (A 1). The direction of the inequality must be reversed when $p > 2$ owing to the convexity of ϕ. Finally, Cocke's (1969) result

$$\tfrac{1}{6}\langle \log (w_1 w_2 w_3)\rangle \leqslant \left\langle \log\left(\frac{l}{l_0}\right)\right\rangle \tag{14}$$

can be established in a similar way using the concave function $\phi = \log (x)$.

Note that since all of the functions involved are strictly increasing, the inequalities in (6)–(14) are strict unless $w_1 = w_2 = w_3$. This occurs only when $\boldsymbol{W}$ is the identity matrix (rotation of the fluid element without distortion) and this must occur with probability one to have equalities in (8), (9), and (11)–(14). This is certainly not meaningful for turbulent flow, and the inequalities that involve ensemble averages are therefore all strict. When the isotropic turbulence is also incompressible ($w_1 w_2 w_3 = 1$),

$$\left\langle \log\frac{l}{l_0}\right\rangle > 0 \quad \text{and} \quad \left\langle \frac{l^p}{l_0^p}\right\rangle > 1, \tag{15}$$

for any $p > 0$. The second inequality in (15) (Orszag 1970) can be verified by either using the first inequality in

$$\left\langle \frac{l^p}{l_0^p}\right\rangle = \exp\left[\log\left\langle \frac{l^p}{l_0^p}\right\rangle\right] \geqslant \exp\left[\left\langle \log\frac{l^p}{l_0^p}\right\rangle\right] = \exp\left[p\left\langle \log\frac{l}{l_0}\right\rangle\right] > 1, \tag{16}$$

or using the arithmetic geometric-mean inequality $\frac{1}{3}(w_1^r + w_2^r + w_3^r) \geqslant ((w_1 w_2 w_3)^r)^{\frac{1}{3}}$ directly to achieve the same end.

3. Curve stretching

In this section we extend the above results to the average deformation of a material curve. The inequality (A 3) used to carry this out is given in the Appendix. Let $\boldsymbol{C}(s;t_0)$ be a parametric representation, for s in the interval $[0, 1]$, of a material curve at time t_0, $\boldsymbol{C}(s;t) = \boldsymbol{x}(\boldsymbol{C}(s;t_0), t)$ be the corresponding curve at time t, and C_0 and C be their arclengths. Define

$$\alpha = \left\langle \frac{w_1^{\frac{1}{2}} + w_2^{\frac{1}{2}} + w_3^{\frac{1}{2}}}{3}\right\rangle \leqslant \beta = \left\langle \left(\frac{w_1 + w_2 + w_3}{3}\right)^{\frac{1}{2}}\right\rangle \leqslant \gamma = \left\langle \frac{w_1 + w_2 + w_3}{3}\right\rangle^{\frac{1}{2}}, \tag{17}$$

where the inequalities are the consequence of (A 2) and (A 1) respectively. Note that in homogeneous turbulence $\boldsymbol{x}(\boldsymbol{a}, t) - \boldsymbol{a}$ is statistically homogeneous in space, Monin & Yaglom (1971, p. 572), and therefore any function of the partial derivatives $\boldsymbol{x}_{,i}(\boldsymbol{a}, t)$ is also statistically homogeneous in space. Consequently, for isotropic turbulence, ensemble averages of functions of w_i, such as α, β, and γ depend only on time. Bounds for the average arclengths in an isotropic incompressible flow can be written in terms of these quantities:

$$C_0 \leqslant \exp\langle \log C\rangle \leqslant \langle C\rangle \leqslant \gamma C_0, \tag{18}$$

$$C_0 \leqslant \alpha C_0 \leqslant \langle C\rangle \leqslant \tfrac{3}{2}\alpha C_0, \tag{19}$$

$$\tfrac{1}{2}\surd 3\,\beta C_0 \leqslant \langle C\rangle \leqslant \surd 3\,\beta C_0. \tag{20}$$

The derivation of these bounds goes as follows: since we can multiply $\boldsymbol{l}_0$ by any constant, no matter how small, without affecting the ratio l/l_0, we let $\boldsymbol{l}_0 = \mathrm{d}\boldsymbol{C}(s;t_0)/\mathrm{d}s = \boldsymbol{C}'(s;t_0)$. Then $\boldsymbol{l} = \boldsymbol{x}_{,k}(\boldsymbol{C}(s,t_0), t)\, C_k'(s, t_0)$ and the first inequality in (15) implies that

$$\langle \log |\boldsymbol{x}_{,k}\, C_k'(s;t_0)|\rangle = \langle \log l\rangle \geqslant \log l_0 = \log |\boldsymbol{C}'(s;t_0)|. \tag{21}$$

Consequently the first inequality in (18) follows from

$$\begin{aligned}
\exp\langle\log C\rangle &= \exp\left\langle\log\left[\int_0^1\left|\frac{\partial \boldsymbol{C}(s;t)}{\partial s}\right|\mathrm{d}s\right]\right\rangle \\
&\geqslant \int_0^1 \exp\left\langle\log\left|\frac{\partial \boldsymbol{C}(s;t)}{\partial s}\right|\right\rangle \mathrm{d}s \quad (\text{by (A 3) } S_1 = [0,1]) \\
&= \int_0^1 \exp\langle\log|\boldsymbol{x}_{,k}C'_k(s;t_0)|\rangle\,\mathrm{d}s \\
&\geqslant \int_0^1 \exp\{\log|\boldsymbol{C}'(s;t_0)|\}\,\mathrm{d}s \quad (\text{by (21)}) \\
&= \int_0^1 |\boldsymbol{C}'(s;t_0)|\,\mathrm{d}s = C_0.
\end{aligned} \tag{22}$$

The second inequality in (18) is an immediate consequence of (A 1) with $\phi = \exp(x)$, and the third follows from (12) by virtue of (A 1) with $\phi = x^{\frac{1}{2}}$. The remaining inequalities follow simply from (8) and (9).

Inequalities (11)–(13) give 'tight' upper and lower bounds for $\langle\int_0^1|\partial \boldsymbol{C}(s;t)/\partial s|^p\,\mathrm{d}s\rangle$ in an obvious way and we use these to get bounds for the moments

$$\left\langle\left(\int_0^1|\partial \boldsymbol{C}(s;t)/\partial s|\,\mathrm{d}s\right)^p\right\rangle \quad \text{for} \quad p>0.$$

Let
$$\alpha_p = \tfrac{1}{3}\langle w_1^{p/2}+w_2^{p/2}+w_3^{p/2}\rangle,$$

then for an isotropic incompressible turbulent flow we can show that for $p>0$

$$C_0^p \leqslant \langle C^p\rangle. \tag{23}$$

We can also show that for $0<p\leqslant 1$

$$\alpha_p \min\{|\boldsymbol{C}'(s;t_0)|^{p-1}\}\,C_0 \leqslant \langle C^p\rangle \leqslant \langle C\rangle^p \leqslant c_2^p C_0^p, \tag{24}$$

and for $p \geqslant 1$

$$C_0^p \leqslant c_1^p C_0^p \leqslant \langle C\rangle^p \leqslant \langle C^p\rangle \leqslant 3\alpha_p \max\{|\boldsymbol{C}'(s;t_0)|^{p-1}\}\,C_0, \tag{25}$$

where c_2 is the minimum of the coefficients of C_0 in the upper bounds for $\langle C\rangle$ in (18)–(20), c_1 is the maximum of the coefficients in the lower bounds, and the min and max functions are taken over s in the interval $[0, 1]$. The bound (23) is established by raising the left-hand side of (18) to the power p, taking p inside the log, and then using (A 1) with $\phi = \exp(x)$. Next we note that the right-hand side of (5) is always less than or equal to $w_1+w_2+w_3$ so that $l^p/l_0^p \leqslant w_1^{p/2}+w_2^{p/2}+w_3^{p/2}$ for $0<p\leqslant 2$. Then we can replace the left-hand side of (11) by $3\alpha_p$ and use (A 1) with $\phi = x^p$ and (11) to (13) to get the leftmost inequality of (24) and the rightmost inequality of (25). The remaining inequalities follow from (18)–(20).

These results enable us to obtain bounds for moments of the length of an initially straight line segment, and for moments of the dispersion of two points, in a turbulent flow. Consider the special case of an initially straight material line $\boldsymbol{C}(s;t_0) = \boldsymbol{a}_1+s(\boldsymbol{a}_2-\boldsymbol{a}_1)$, and note that $|\boldsymbol{C}'(s;t_0)| = |\boldsymbol{a}_2-\boldsymbol{a}_1| \equiv L_0$ is independent of s. It follows from (23)–(25) that for $p>0$

$$d_1 L_0^p \leqslant \langle C^p\rangle \leqslant d_2 L_0^p, \tag{26}$$

$$\langle|\boldsymbol{x}(\boldsymbol{a}_2,t)-\boldsymbol{x}(\boldsymbol{a}_1,t)|^p\rangle \leqslant d_2 L_0^p, \tag{27}$$

where $d_1 = \min[c_1^p, \alpha_p]$ and $d_2 = \max[c_2^p, 3\alpha_p]$ with c_1, c_2 and α_p defined as in (24) and (25).

4. Surface stretching

The results for the line stretching can be extended to the stretching of a material surface, just as in the work of Cocke (1969) and Orszag (1970). Only minor modifications are required.

Let $\boldsymbol{l}_0$ and $\boldsymbol{k}_0$ be two infinitesimal line elements at $t = t_0$ and form an infinitesimal surface by taking their vector product $\boldsymbol{S}_0 = \boldsymbol{l}_0 \times \boldsymbol{k}_0$. At time t the surface element is deformed into

$$\boldsymbol{S} = \boldsymbol{l} \times \boldsymbol{k} = (\boldsymbol{x}_{,i} l_{0i}) \times (\boldsymbol{x}_{,j} k_{0j}). \tag{28}$$

Let the deformation tensor $\boldsymbol{W}$, its eigenvalues w_i, and the rotation matrix $\boldsymbol{A}$ be defined as before. Use of the vector identity

$$|\boldsymbol{a} \times \boldsymbol{b}|^2 = |\boldsymbol{a}|^2 |\boldsymbol{b}|^2 - (\boldsymbol{a} \cdot \boldsymbol{b})^2 \tag{29}$$

and $S^2 = |\boldsymbol{A}\boldsymbol{l}_0 \times \boldsymbol{A}\boldsymbol{k}_0|^2$ gives the stretching of a material surface element by one realization of a velocity field,

$$\begin{aligned} \frac{S^2}{S_0^2} &= w_2 w_3 \frac{(\boldsymbol{A}\boldsymbol{l}_0 \times \boldsymbol{A}\boldsymbol{k}_0)_1^2}{|\boldsymbol{A}\boldsymbol{l}_0 \times \boldsymbol{A}\boldsymbol{k}_0|^2} + w_1 w_3 \frac{(\boldsymbol{A}\boldsymbol{l}_0 \times \boldsymbol{A}\boldsymbol{k}_0)_2^2}{|\boldsymbol{A}\boldsymbol{l}_0 \times \boldsymbol{A}\boldsymbol{k}_0|^2} + w_1 w_2 \frac{(\boldsymbol{A}\boldsymbol{l}_0 \times \boldsymbol{A}\boldsymbol{k}_0)_3^2}{|\boldsymbol{A}\boldsymbol{l}_0 \times \boldsymbol{A}\boldsymbol{k}_0|^2}, \\ &= w_2 w_3 \sin^2\theta \cos^2\psi + w_1 w_3 \sin^2\theta \sin^2\psi + w_1 w_2 \cos^2\theta, \end{aligned} \tag{30}$$

where θ and ψ are spherical polar angles of the vector $\boldsymbol{l}_0 \times \boldsymbol{k}_0$ relative to the principal axes of the symmetric tensor that arises when S^2 is formed by squaring (28). This matrix is analogous to $\boldsymbol{W}$ and can be written in terms of it. Equation (30) is analogous to (5) and we need only to replace the eigenvalues w_1, w_2, w_3 by the eigenvalues $w_2 w_3, w_3 w_1, w_1 w_2$; the line elements $\boldsymbol{l}, \boldsymbol{l}_0$ by the surface elements $\boldsymbol{S}, \boldsymbol{S}_0$; the arclengths C, C_0 by the surface areas S, S_0; and the line interval $[0, 1]$ by the surface domain $\mathscr{D}$ in the line-stretching results to obtain the new ones for surface stretching. Here $\boldsymbol{S}(u, v; t_0)$ is the parametric representation at the initial time t_0 of the material surface. Its area, which is initially S_0, becomes

$$S = \iint_{\mathscr{D}} \left| \frac{\partial \boldsymbol{x}(\boldsymbol{S}(u, v; t_0), t)}{\partial u} \times \frac{\partial \boldsymbol{x}(\boldsymbol{S}(u, v; t_0), t)}{\partial v} \right| du\, dv, \tag{31}$$

at time t.

The only *non-symbolic* modifications of our analysis are:

(*a*) if the area of $\mathscr{D}$ is not one in (23)–(25), the right-hand side of (25) and the left-hand side of (24) must be multiplied by the area of $\mathscr{D}$ to the power $p-1$ for correct usage of Jensen's inequality;

(*b*) in (25) and (27) the notion of a distance between two points has to be replaced by the projected area of a region onto a plane and the left-hand side of (27) has to be interpreted correctly.

5. Summary

In a turbulent flow one expects the average length (area) of a finite material line (surface) to stretch, in comparison with its initial value, in the course of its time evolution. We have shown that this is indeed the case for isotropic incompressible turbulence. Moreover, we have also obtained upper and lower bounds for all the moments of the amount of stretch in terms of the eigenvalues of the Green deformation tensor. The results have been subsequently used to give an upper bound for all the moments of the relative dispersion of two fluid particles in terms of their initial separation. This work, to some extent, uses and completes the work of Cocke (1969) who basically shows that the moments of infinitesimal material line (surface) elements only increase in time.

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Appendix.

(*Jensen's inequality*). Let X be a random variable and ϕ a convex (concave) function containing the range of X. Assume that both X and $\phi(X)$ have ensemble averages, then

$$\phi(\langle X\rangle) \leqslant (\geqslant) \langle\phi(X)\rangle. \tag{A 1}$$

For a proof, see any textbook on probability theory or measure theory, e.g. Billingsley (1986, p. 283).

The following special case of Jensen's inequality has been used frequently in this work. Let p_1, p_2, and p_3 be three positive numbers that sum to one, and let a_1, a_2, and a_3 be any real numbers. Then for convex (concave) functions ϕ as above we have

$$\phi\left(\sum_{i=1}^{3} p_i a_i\right) \leqslant (\geqslant) \sum_{i=1}^{3} p_i \phi(a_i). \tag{A 2}$$

This is obtained by taking X in (A 1) as a random variable that takes the value a_i with probability p_i for the ensemble $i = 1, 2, 3$.

Let $Y(t)$ be a random process. Then under minimum regularity conditions

$$\int_{S_1} \exp\langle\log|Y(s_1)|\rangle \, \mathrm{d}s_1 \leqslant \exp\left\langle\log\int_{S_1}|Y(s_1)|\,\mathrm{d}s_1\right\rangle. \tag{A 3}$$

This is the consequence of the following more general result that can be found in Dunford & Schwartz (1958, p. 535).

Let (S, Σ, μ) and (S_1, Σ_1, μ_1) be positive measure spaces. Assume $\mu(S) = 1$. Then if K is $\mu\times\mu_1$-measurable function defined on $S\times S_1$,

$$\int_{S_1}\exp\left\{\int_S \log|K(s,s_1)|\mu(\mathrm{d}s)\right\}\mu_1(\mathrm{d}s_1) \leqslant \exp\left\{\int_S \log\left[\int_{S_1}|K(s,s_1)|\mu_1(\mathrm{d}s_1)\right]\mu(\mathrm{d}s)\right\}. \tag{A 4}$$

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